Fourier Transforms

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Contents

In the study of Fourier series, we have seen that if a function $f(x)$ is defined in $-\infty < x < \infty$ and is periodic with period $2l$ satisfy the Dirichlet's conditions viz. (i) $f(x)$ is piecewise continuous and (ii) $f(x)$ has a finite number of maxima and minima, then $f(x)$ can be expanded as a Fourier series as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$. The series on the right hand side converges to $f(x)$ at all points where $f(x)$ is continuous and converges to $\frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$ at points of discontinuity. But many applied problems give rise to non-periodic functions. If a function $f(x)$ is initially defined on a finite interval say, $c \leq x \leq c+2l$, we can always extend the definitions outside $[c, c+2l]$ by imposing some sort of periodicity conditions. If $f(x)$ is defined in $(-\infty, \infty)$ and is non-periodic, we cannot expand $f(x)$ as a Fourier series. Therefore if we think of $f(x)$ as a periodic function with infinite period, f and f' are piecewise continuous on every finite interval $[-l, l]$ and if $\int_{-\infty}^{\infty} |f(x)|dx < \infty$, then one can extend the concept of Fourier series to this function and obtain a representation as an indefinite integral, called Fourier integral.

1 FOURIER INTEGRAL

Let f and f' be piecewise continuous functions defined on every finite interval $[-l, l]$ and let

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

i.e. $f(x)$ is *absolutely integrable*. Then the ***Fourier integral*** of f is given by

$$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]dw \tag{1}$$

for all points x at which f is continuous. The integral converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ for every point x at which f is discontinuous, where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\, dt \tag{2}$$

and

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\, dt. \tag{3}$$

Theorem 1.1 (Fourier Integral)

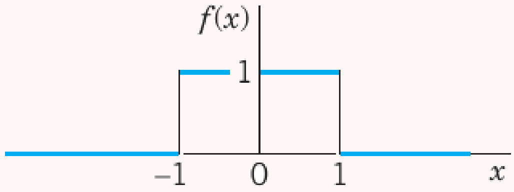
If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if f is absolutely integrable, then $f(x)$ can be represented by a Fourier integral (1) with A and B given by (2) and (3). At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left-and right-hand limits of $f(x)$ at that point.

Problem 1.1

Express the function

$$f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

as a Fourier integral. Hence evaluate $\displaystyle\int_0^\infty \frac{\sin w \cos wx}{w} dw$.



Solution. The Fourier integral of $f(x)$ is

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx] dw.$$

We have

$$\begin{aligned} A(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\, dt \\ &= \frac{1}{\pi}\int_{-1}^{1} 1 \cos wt dt \\ &= \frac{1}{\pi}\left[\frac{\sin wt}{w}\right]_{t=-1}^{1} \\ &= \frac{1}{\pi w}[\sin w - \sin(-w)] \\ &= \frac{1}{\pi w}[\sin w + \sin w] \\ &= \frac{2\sin w}{\pi w}. \end{aligned}$$

Similarly,

$$\begin{aligned} B(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\, dt \\ &= \frac{1}{\pi}\int_{-1}^{1} 1 \sin wt dt \\ &= 0. \end{aligned}$$

Hence the Fourier integral of f is given by

$$\begin{aligned} f(x) &= \int_0^\infty [A(w)\cos wx + B(w)\sin wx] dw \\ &= \int_0^\infty \left[\frac{2\sin w}{\pi w}\cos wx + 0\right] dw \\ &= \int_0^\infty \frac{2\sin w \cos wx}{\pi w} dw \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sin w \cos wx}{w} dw, \end{aligned}$$

for $x \neq \pm 1$, since f is continuous for all $x \neq \pm 1$. For $x = 1$, the right and left hand limits of f are given by

$$f(1^+) = 0 \quad \text{and} \quad f(1^-) = 1.$$

The average of these limits is $\frac{0+1}{2} = \frac{1}{2}$. Hence for $x = 1$ we get

$$\frac{2}{\pi}\int_0^\infty \frac{\sin w \cos w}{w} dw = \frac{1}{2}.$$

For $x = -1$, the right and left hand limits of f are given by

$$f(-1^+) = 1 \quad \text{and} \quad f(-1^-) = 0.$$

The average of these limits is $\frac{1+0}{2} = \frac{1}{2}$. Hence for $x = -1$ also, we get

$$\frac{2}{\pi}\int_0^\infty \frac{\sin w \cos w}{w} dw = \frac{1}{2}.$$

Thus

$$\begin{aligned} \int_0^\infty \frac{\sin w \cos wx}{w} dw &= \frac{\pi}{2} f(x) \\ &= \frac{\pi}{2}\begin{cases} 1 & \text{for } |x| < 1 \\ \frac{1}{2} & \text{for } x = \pm 1 \\ 0 & \text{for } |x| > 1. \end{cases} \\ &= \begin{cases} \pi/2 & \text{for } |x| < 1 \\ \pi/4 & \text{for } x = \pm 1 \\ 0 & \text{for } |x| > 1. \end{cases} \end{aligned}$$

■

▶ The above problem shows that

$$\int_0^\infty \frac{\sin t \cos t}{t} dt = \frac{\pi}{4}$$

and

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

(put $x = 0$ in the above problem).

Problem 1.2

Show that

$$\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}\,dx = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0. \end{cases}$$

Solution. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ e^{-x} & \text{if } x > 0. \end{cases}$$

We have

$$\int e^{-st}\sin wt\,dt = \frac{e^{-st}}{s^2+w^2}(-w\cos wt - s\sin wt)$$
$$\int e^{-st}\cos wt\,dt = \frac{e^{-st}}{s^2+w^2}(-s\cos wt + w\sin wt)$$

We represent f as a Fourier integral. The Fourier integral of $f(x)$ is

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]dw.$$

We have

$$\begin{aligned}
A(w) &= \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos wt\,dt \\
&= \frac{1}{\pi}\int_0^\infty e^{-t}\cos wt dt \\
&= \frac{1}{\pi}\lim_{k\to\infty}\int_0^k e^{-t}\cos wt\,dt \\
&= \frac{1}{\pi}\lim_{k\to\infty}\left[\frac{e^{-t}}{1+w^2}(-\cos wt + w\sin wt)\right]_0^k \\
&= \frac{1}{\pi}\lim_{k\to\infty}\left[\frac{e^{-k}}{1+w^2}(-\cos wk + w\sin wk) - \frac{e^0}{1+w^2}(-\cos 0 + w\sin 0)\right] \\
&= \frac{1}{\pi}\lim_{k\to\infty}\left[\frac{e^{-k}}{1+w^2}(-\cos wk + w\sin wk) - \frac{1}{1+w^2}(-1+0)\right] \\
&= \frac{1}{\pi}\left[0 + \frac{1}{1+w^2}\right] \\
&= \frac{1}{\pi(1+w^2)}
\end{aligned}$$

Similarly,

$$
\begin{aligned}
B(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\,dt \\
&= \frac{1}{\pi}\int_{0}^{\infty} \mathrm{e}^{-t}\sin wt dt \\
&= \frac{1}{\pi}\lim_{k\to\infty}\int_{0}^{k} \mathrm{e}^{-t}\sin wt\,dt \\
&= \frac{1}{\pi}\lim_{k\to\infty}\left[\frac{\mathrm{e}^{-t}}{1+w^2}(-w\cos wt-\sin wt)\right]_0^k \\
&= \frac{1}{\pi}\lim_{k\to\infty}\left[\frac{\mathrm{e}^{-k}}{1+w^2}(-w\cos wk-\sin wk)-\frac{\mathrm{e}^{0}}{1+w^2}(-w\cos 0-\sin 0)\right] \\
&= \frac{1}{\pi}\lim_{k\to\infty}\left[\frac{\mathrm{e}^{-k}}{1+w^2}(-w\cos wk-\sin wk)-\frac{1}{1+w^2}(-w+0)\right] \\
&= \frac{1}{\pi}\left[0+\frac{w}{1+w^2}\right] \\
&= \frac{w}{\pi(1+w^2)}.
\end{aligned}
$$

Hence the Fourier integral of f is given by

$$
\begin{aligned}
f(x) &= \int_0^\infty [A(w)\cos wx + B(w)\sin wx]dw \\
&= \int_0^\infty \left[\frac{1}{\pi(1+w^2)}\cos wx + \frac{w}{\pi(1+w^2)}\sin wx\right]dw \\
&= \frac{1}{\pi}\int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}\,dw,
\end{aligned}
$$

for $x \neq 0$, since f is continuous for all $x \neq 0$. For $x = 0$, the right and left hand limits of f are given by

$$f(0^+) = \mathrm{e}^{-0} = 1 \quad \text{and} \quad f(0^-) = 0.$$

The average of these limits is $\frac{1+0}{2} = \frac{1}{2}$. Hence for $x = 0$ we get

$$\frac{1}{\pi}\int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}\,dw = \frac{1}{2}.$$

Thus

$$\int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}\,dw = \pi f(x)$$
$$= \pi \begin{cases} 0 & \text{if } x<0 \\ \frac{1}{2} & \text{if } x=0 \\ e^{-x} & \text{if } x>0. \end{cases}$$
$$= \begin{cases} 0 & \text{if } x<0 \\ \frac{\pi}{2} & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0. \end{cases}$$

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FOURIER COSINE INTEGRAL AND FOURIER SINE INTEGRAL

Just as Fourier series simplify if a function is even or odd, so do Fourier integrals, and you can save work. Indeed, if f has a Fourier integral representation and is even, then $B(w)=0$. Then (1) reduces to a ***Fourier cosine integral***

$$f(x) = \int_0^\infty A(w)\cos wx\,dw \quad \text{where} \quad A(w) = \frac{2}{\pi}\int_0^\infty f(t)\cos wt\,dt \tag{4}$$

Similarly, if f has a Fourier integral representation and is odd, then $A(w)=0$. Then (1) becomes a ***Fourier sine integral***

$$f(x) = \int_0^\infty B(w)\sin wx\,dw \quad \text{where} \quad B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt\,dt \tag{5}$$

Problem 1.3

Express

$$f(x) = \begin{cases} 1/2 & \text{for } 0 \le x \le \pi \\ 0 & \text{for } x > \pi \end{cases}$$

as a Fourier sine integral and hence show that

$$\int_0^\infty \frac{1-\cos \pi w}{w}\sin xw\,dw = \begin{cases} \frac{\pi}{2} & \text{for } 0<x<\pi \\ 0 & \text{for } x>\pi \end{cases}$$

Solution. The Fourier sine integral of $f(x)$ is

$$f(x) = \int_0^\infty B(w)\sin wx\,dw.$$

We have

$$\begin{aligned}
B(w) &= \frac{2}{\pi}\int_0^\infty f(t)\sin wt\,dt \\
&= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\sin wt dt \\
&= \frac{2}{2\pi}\left[\frac{-\cos wt}{w}\right]_{t=0}^{\pi} \\
&= \frac{-1}{\pi w}[\cos w\pi - \cos(0)] \\
&= \frac{-1}{\pi w}[\cos w\pi - 1] \\
&= \frac{1-\cos w\pi}{\pi w}.
\end{aligned}$$

The Fourier sine integral of $f(x)$ is

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{1-\cos \pi w}{\pi w}\cdot \sin wx\,dw \\
&= \frac{1}{\pi}\int_0^\infty \frac{1-\cos \pi w}{w}\cdot \sin wx\,dw
\end{aligned}$$

$$\begin{aligned}
\therefore \int_0^\infty \frac{1-\cos \pi w}{w}\sin wx\,dw &= \pi f(x) \\
&= \begin{cases} \pi/2 & \text{for } 0 < x < \pi \\ 0 & \text{for } x > \pi \end{cases}
\end{aligned}$$

At $x = \pi$, the average of left and right hand limits is $\frac{1}{2}(\frac{1}{2}+0) = \frac{1}{4}$. Hence at $x = \pi$,

$$\int_0^\infty \frac{1-\cos \pi w}{w}\sin w\pi\,dw = \frac{\pi}{4}.$$

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Problem 1.4

Using Fourier integrals, show that

$$\int_0^\infty \frac{w\sin wx}{k^2+w^2}dw = \frac{\pi}{2}e^{-kx}\ (x>0, k>0).$$

Solution. Let $f(x) = e^{-kx}$ where $x > 0$ and $k > 0$. Using Fourier sine integral, we have

$$f(x) = \int_0^\infty B(w)\sin wx\,dw.$$

where

$$
\begin{aligned}
B(w) &= \frac{2}{\pi}\int_0^\infty f(t)\sin wt\,dt \\
&= \frac{2}{\pi}\int_0^\infty e^{-kt}\sin wt\,dt \\
&= \frac{2}{\pi}\lim_{p\to\infty}\left[\frac{e^{-kt}}{k^2+w^2}(-k\sin wt - w\cos wt)\right]_{t=0}^{p} \\
&= \frac{2}{\pi}\lim_{p\to\infty}\left[\frac{e^{-kp}}{k^2+w^2}(-k\sin wp - w\cos wp) - \frac{e^0}{k^2+w^2}(-k\sin 0 - w\cos 0)\right] \\
&= \frac{2}{\pi}\left[0+\frac{1}{k^2+w^2}(w)\right] \\
&= \frac{2}{\pi}\frac{w}{k^2+w^2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int_0^\infty \frac{2}{\pi}\frac{w\sin wx}{k^2+w^2}dw &= f(x) \\
\Rightarrow \int_0^\infty \frac{w\sin wx}{k^2+w^2}dw &= \frac{\pi}{2}f(x) \\
&= \frac{\pi}{2}e^{-kx}.
\end{aligned}
$$

■

Problem 1.5

Using Fourier integral, prove that

$$
\int_0^\infty \frac{\cos(\pi w/2)\cos xw}{1-w^2}dw = \begin{cases} \frac{\pi}{2}\cos x & \text{if } |x| < \pi/2 \\ 0 & \text{if } |x| > \pi/2 \end{cases}
$$

Solution. Let

$$
f(x) = \begin{cases} \cos x & \text{if } |x| < \pi/2 \\ 0 & \text{if } |x| > \pi/2. \end{cases}
$$

The Fourier cosine integral of $f(x)$ is

$$
f(x) = \int_0^\infty A(w)\cos wx\,dw.
$$

where

$$
\begin{aligned}
A(w) &= \frac{2}{\pi}\int_0^\infty f(t)\cos wt\,dt \\
&= \frac{2}{\pi}\int_0^{\pi/2} \cos t\cdot\cos wt\,dt \\
&= \frac{2}{\pi}\left[\int_0^{\pi/2}\frac{1}{2}[\cos(w+1)t+\cos(w-1)t\right]dt \\
&= \frac{1}{\pi}\left[\frac{\sin(w+1)t}{w+1}+\frac{\sin(w-1)t}{w-1}\right]_{t=0}^{\pi/2} \\
&= \frac{1}{\pi}\left[\frac{\sin(w+1)\pi/2}{w+1}+\frac{\sin(w-1)\pi/2}{w-1}\right] \\
&= \frac{1}{\pi}\left[\frac{\sin w(\pi/2)\cos\frac{\pi}{2}+\cos w\frac{\pi}{2}\sin\frac{\pi}{2}}{w+1}+\frac{\sin w\frac{\pi}{2}\cdot\cos\frac{\pi}{2}-\cos w\frac{\pi}{2}\sin\frac{\pi}{2}}{w-1}\right] \\
&= \frac{1}{\pi}\cos w\frac{\pi}{2}\left[\frac{1}{w+1}-\frac{1}{w-1}\right] \\
&= \frac{1}{\pi}\cos\left(\frac{w\pi}{2}\right)\left(\frac{-2}{w^2-1}\right) \\
&= \frac{2\cos(w\pi/2)\cos wx}{\pi(1-w^2)}.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int_0^\infty \frac{2\cos(w\pi/2)\cos wx}{\pi(1-w^2)}dw &= f(x) \\
\Rightarrow \int_0^\infty \frac{\cos(w\pi/2)\cos wx}{1-w^2}dw &= \frac{\pi}{2}f(x) \\
&= \begin{cases} (\pi/2)\cos x, & |x|<\pi/2 \\ 0, & |x|>\pi/2 \end{cases}
\end{aligned}
$$

Consider the case of $|x| = \frac{\pi}{2}$. When $x = \frac{\pi}{2}$, the average of left and right hand limits is $\frac{1}{2}(\cos\frac{\pi}{2} + 0) = 0$. So

$$\int_0^\infty \frac{\cos(w\pi/2)\cos wx}{1-w^2}dw = 0$$

for $x = \frac{\pi}{2}$. Similarly for $x = -\frac{\pi}{2}$, the average of left and right hand limits is $\frac{1}{2}(0+\cos\frac{\pi}{2}) = 0$ and hence

$$\int_0^\infty \frac{\cos(w\pi/2)\cos wx}{1-w^2}dw = 0$$

for $x = -\frac{\pi}{2}$. ■

2 FOURIER COSINE AND SINE TRANSFORMS

The ***Fourier cosine transform*** of $f(x)$ is given by

$$\hat{f}_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx. \tag{6}$$

and the ***inverse Fourier cosine transform*** of $\hat{f}_c(s)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(s) \cos sx\, ds. \tag{7}$$

Similarly, the ***Fourier sine transform*** of $f(x)$ is given by

$$\hat{f}_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx. \tag{8}$$

and the ***inverse Fourier sine transform*** of $\hat{f}_s(s)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(s) \sin sx\, ds. \tag{9}$$

Problem 2.1

Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

Solution. We have

$$\begin{aligned}
\hat{f}_c(s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^a k \cos sx\, dx \\
&= \sqrt{\frac{2}{\pi}} k \left[\frac{\sin sx}{s} \right]_0^a \\
&= \sqrt{\frac{2}{\pi}} \frac{k}{s} [\sin sa - \sin 0] \\
&= \sqrt{\frac{2}{\pi}} \frac{k \sin sa}{s}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\hat{f}_s(s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^a k \sin sx\, dx \\
&= \sqrt{\frac{2}{\pi}} k \left[\frac{-\cos sx}{s} \right]_0^a \\
&= \sqrt{\frac{2}{\pi}} \frac{-k}{s} [\cos sa - \cos 0] \\
&= \sqrt{\frac{2}{\pi}} \frac{k(1 - \cos sa)}{s}.
\end{aligned}$$

■

PROPERTIES

1. $\hat{f}_c[\alpha f(x)+\beta g(x)] = \alpha \hat{f}_c[f(x)] + \beta \hat{f}_c[g(x)]$ and

   $\hat{f}_s[\alpha f(x)+\beta g(x)] = \alpha \hat{f}_s[f(x)] + \beta \hat{f}_s[g(x)]$.

2. $\hat{f}_c[f(ax)] = \frac{1}{a}\hat{f}_c(s/a)$ and $\hat{f}_s[f(ax)] = \frac{1}{a}\hat{f}_s(s/a)$.

Proof. We have

$$\hat{f}_c[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cdot \cos sx\,dx.$$

Put $ax = t$. Then $adx = dt$.

$$\begin{aligned}
\therefore \hat{f}_c[f(ax)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cdot \cos s(t/a)\frac{dt}{a} \\
&= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\left(\frac{s}{a}\right)t\,dt \\
&= \frac{1}{a}\hat{f}_c(s/a).
\end{aligned}$$

The other result is similar. ∎

Assume that Fourier sine and cosine transforms of $f(x)$ exist and let $f(x)\to 0$ as $x\to\infty$. Then

3. $\hat{f}_c[f'(x)] = s\hat{f}_s(s) - \sqrt{\frac{2}{\pi}}f(0)$ and

   $\hat{f}_s[f'(x)] = -s\hat{f}_c(s)$

Proof. We have

$$\begin{aligned}
\hat{f}_c[f'(x)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos sx\,\mathrm{dx} \\
&= \sqrt{\frac{2}{\pi}}\lim_{k\to\infty}[\cos sx\cdot f(x)]_0^k - \sqrt{\frac{2}{\pi}}\int_0^\infty(-s\sin sx)\cdot f(x)\,dx \\
&= \sqrt{\frac{2}{\pi}}\lim_{k\to\infty}[\cos sk\cdot f(k) - \cos 0 f(0)] - \sqrt{\frac{2}{\pi}}\int_0^\infty(-s\sin sx)\cdot f(x)\,dx \\
&= 0 - \sqrt{\frac{2}{\pi}}f(0) + s\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx \\
&= s\hat{f}_s[f(x)] - \sqrt{\frac{2}{\pi}}f(0).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\hat{f}_s[f'(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin sx \, dx \\
&= \sqrt{\frac{2}{\pi}} \lim_{k\to\infty} [\sin sx \cdot f(x)]_0^k - \sqrt{\frac{2}{\pi}} \int_0^\infty s \cos sx \cdot f(x) \, dx \\
&= \sqrt{\frac{2}{\pi}} \lim_{k\to\infty} [\sin sk \cdot f(k) - \sin 0 f(0)] - \sqrt{\frac{2}{\pi}} \int_0^\infty s \cos sx \cdot f(x) \, dx \\
&= 0 - s\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \\
&= -s\hat{f}_c[f(x)].
\end{aligned}$$

■

3 FOURIER TRANSFORMS

We derived two real transforms, Fourier sine and cosine. Now we want to derive a complex transform that is called the Fourier transform. It will be obtained from the complex Fourier integral, which will be discussed next.

COMPLEX FORM OF THE FOURIER INTEGRAL

From the definition of Fourier integral from equations (1), (2), and (3), we have

$$\begin{aligned}
f(x) &= \int_0^\infty \left[\left(\frac{1}{\pi} \int_{-\infty}^\infty f(t) \cos wt \, dt \right) \cos wx + \left(\frac{1}{\pi} \int_{-\infty}^\infty f(t) \sin wt \, dt \right) \sin wx \right] dw \\
&= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty (\cos wt \cos wx + \sin wt \sin wx) f(t) dt \right] dw \\
\Rightarrow f(x) &= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos w(x-t) dt \, dw. \qquad (10)
\end{aligned}$$

Since $\cos w(x-t) = \frac{1}{2}\left[e^{iw(x-t)} + e^{-iw(x-t)}\right]$, equation (10) becomes

$$\begin{aligned}
f(x) &= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \tfrac{1}{2} \left[e^{iw(x-t)} + e^{-iw(x-t)}\right] dt \, dw \\
&= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty f(t) e^{iw(x-t)} dt \, dw + \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty f(t) e^{-iw(x-t)} dt \, dw.
\end{aligned}$$

To combine the two terms on the RHS, we change the dummy integration variable from w to $-w$ in the second. Thus

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty f(t) e^{iw(x-t)} dt \, dw + \frac{1}{2\pi} \int_0^{-\infty} \int_{-\infty}^\infty f(t) e^{iw(x-t)} dt \, d(-w) \\
&= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty f(t) e^{iw(x-t)} dt \, dw + \frac{1}{2\pi} \int_{-\infty}^0 \int_{-\infty}^\infty f(t) e^{iw(x-t)} dt \, dw \\
\Rightarrow f(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) e^{iw(x-t)} dt \, dw. \qquad (11)
\end{aligned}$$

(11) is known as the complex form of the Fourier integral.

▶ Note that the complex form of Fourier integral can also be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt}\, dt \right] e^{iwx}\, dw$$

This motivates the definition of Fourier transform.

FOURIER TRANSFORM AND ITS INVERSE

The ***Fourier transform*** of $f(x)$ is defined as

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt}\, dt \tag{12}$$

and the ***inverse Fourier transform*** of $\hat{f}(w)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw \tag{13}$$

CONVOLUTION

The convolution of f and g is denoted by $f * g$ and is defined as

$$f * g = \int_0^t f(u) g(t-u)\, du. \tag{14}$$

Also $f * g = g * f$.

Theorem 3.1 (Convolution Theorem)

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then the Fourier transform of $f * g$ is given by

$$(f \hat{*} g)(w) = \sqrt{2\pi} \hat{f}(w) \hat{g}(w).$$

The convolution theorem shows that

$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w) \hat{g}(w) e^{iwx}\, dw.$$

Problem 3.1

Find the Fourier transform of

$$f(x) = \begin{cases} x, & |x| < a \\ 0, & |x| > a \end{cases}$$

Solution. Fourier transform of $f(x)$ is given by

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{iwx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x(\cos wx + i \sin wx) dx \\
&= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{a} x \cdot i \sin wx \, dx \\
& \qquad (\because \ x \cos wx \text{ is odd and } x \sin wx \text{ is even}) \\
&= \sqrt{\frac{2}{\pi}} \, i \left\{ x \left(-\frac{\cos wx}{w} \right) - 1 \cdot \left(-\frac{\sin wx}{w^2} \right) \right\}_0^a \\
&= \sqrt{\frac{2}{\pi}} \, i \left[\frac{\sin aw - aw \cos aw}{w^2} \right].
\end{aligned}$$

■

Problem 3.2

Find the Fourier transform of $e^{-x^2/2}$. Show that $e^{-x^2/2}$ is self reciprocal.

Solution. The Fourier transform of $f(x)$ is given by

$$\begin{aligned}
\hat{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} (\cos sx + i \sin sx) dx \\
&= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-x^2/2} \cos sx \, dx \ (\because e^{-x^2/2} \sin sx \text{ is odd}) \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-x^2/2} \cos sx \, dx = I(say) \qquad (*1)
\end{aligned}$$

Now

$$\begin{aligned}
\frac{dI}{ds} &= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-x^2/2} (-\sin sx) \cdot x \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} (\sin sx)(-x e^{-x^2/2} dx) \\
&= \sqrt{\frac{2}{\pi}} \left\{ \left[(\sin sx)(e^{-x^2/2}) \right]_0^{\infty} - \int_{0}^{\infty} s \cos sx \left(e^{-x^2/2} \right) dx \right\} \\
&= \sqrt{\frac{2}{\pi}} \cdot (-s) \int_{0}^{\infty} e^{-x^2/2} \cdot \cos sx \, dx \\
&= -sI.
\end{aligned}$$

Seperating the variables,

$$\frac{dI}{I} = -s \, ds.$$

Integrating, we get

$$\log I = \frac{-s^2}{2} + \log A$$
$$\text{or } I = Ae^{-s^2/2}. \qquad (*2)$$

Now, when $s = 0$, from (*1),

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} dx$$
$$i.e. I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-y^2} \sqrt{2} dy \quad \text{where } y = x/\sqrt{2}$$
$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-y^2} dy = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1.$$

Also when $s = 0$, from (*2), $I = A$. Equating the above values we get $A = 1$.

$$\therefore I = \hat{f}(s) = e^{-s^2/2}$$

Since $\hat{f}(s) = e^{-s^2/2} = f(s), f(x)$ is self reciprocal. ■

EXAMPLE 3.8

Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$ and use it to evaluate $\int_0^\infty \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx$

[K. U. 2000 October, 2001 October]

Solution

Fourier transform of $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{isx} dx.$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1 - x^2)(\cos sx + i \sin sx) dx.$$
$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^1 (1 - x^2)(\cos sx dx \quad (\because (1 - x^2) \sin sx \text{ is odd})$$
$$= \sqrt{\frac{2}{\pi}} \left\{ (1 - x^2) \left(\frac{\sin sx}{s} \right) - (-2x) \left(-\frac{\cos sx}{s^2} \right) + (-2) \left(-\frac{\sin sx}{s^3} \right) \right\}_0^1,$$
$$= \sqrt{\frac{2}{\pi}} \left[-\frac{2 \cos s}{s^2} + \frac{2 \sin s}{s^3} \right].$$
$$= -2\sqrt{\frac{2}{\pi}} \left[\frac{s \cos s - \sin s}{s^3} \right].$$

By inversion formula for Fourier transform,

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -2\sqrt{\frac{2}{\pi}}\left[\frac{s\cos s-\sin s}{s^3}\right](\cos sx - i\sin sx)\,ds \\
&= -\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sxds \\
&\quad +\frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\sin sxds \\
&= -\frac{2}{\pi}2\int_{0}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sxds
\end{aligned}$$

(since the integrand in the second integral is odd)

$$\begin{aligned}
\therefore \int_0^{\infty}\frac{s\cos s-\sin s}{s^3}\cos sxds &= -\frac{\pi}{4}f(x) \\
&= -\frac{\pi}{4}(1-x^2) \text{ if } |x|<1 \\
&= 0 \text{ if } |x|>|
\end{aligned}$$

Put $x = 1/2$.

$$\begin{aligned}
&\int_0^{\infty}\frac{s\cos s-\sin s}{s^3}\cos\frac{s}{2}ds = \frac{-3\pi}{16} \\
\text{or } &\int_0^{\infty}\frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}dx = \frac{-3\pi}{16}.
\end{aligned}$$

3.1 PROPERTIES OF FOURIER TRANSFORMS

P.1. $F[\alpha f(x)+\beta g(x)] = \alpha F[f(x)]+\beta F[g(x)]$.

PROOF

$$\begin{aligned}
F[\alpha f(x)+\beta g(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\alpha f(x)+\beta g(x)]e^{isx}dx \\
&= \alpha\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \\
&\quad +\beta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{isx}dx \\
&= \alpha F[f(x)]+\beta F[g(x)]
\end{aligned}$$

P.2. $F[f(x-a)] = e^{isa}F(s)$.

PROOF

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) \cdot e^{isx} dx$$

$$\text{Put } x-a=t \quad \text{or} \quad x=t+a.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{is(t+a)} \, dt.$$

$$= e^{isa} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{ist} dt$$
$$= e^{isa} F[f(x)] = e^{isa} F(s).$$

P.3. $F[e^{iax} f(x)] = \bar{f}(s+a) = F(s+a)$.

PROOF

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) \cdot e^{isx} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx.$$
$$= \bar{f}(s+a) = F(s+a),$$

P.4. $F[f(ax)] = \frac{1}{|a|} \bar{f}(s/a) = \frac{1}{|a|} F(s/a)$.

PROOF

Case (i) when $a > 0$.

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) \cdot e^{isx} dx \qquad \text{put} \quad ax = t$$
$$adx = dt$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist/a} dt/a$$
$$= \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{i(s/a)t} dt$$
$$= \frac{1}{a} \bar{f}(s/a) = \frac{1}{|a|} F(s/a)$$

Case (ii) when $a < 0$,

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx$$

put $ax = t$ when $x = -\infty$, $t = \infty$

when $x = \infty$, $t = -\infty$ (since $a < 0$).

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(t/a)} \frac{dt}{a}$$

$$= -\frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{it(s/a)}dt$$
$$= -\frac{1}{a}\bar{f}(s/a) = \frac{1}{|a|}F(s/a)$$
$$\therefore \quad F[f(ax)] = \frac{1}{|a|}\bar{f}(s/a).$$

P.5. $F[f(x)\cos ax] = \frac{1}{2}[F(s+a)+F(s-a)]$

PROOF

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax \cdot e^{isx}dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot\left[\frac{e^{iax}+e^{-iax}}{2}\right]e^{isx}dx$$
$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left[e^{i(s+a)x}+e^{i(s-a)x}\right]dx$$
$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s-a)x}dx\right]$$
$$= \frac{1}{2}[F(s+a)+F(s-a)]$$

P.6. $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n}F(s).$

PROOF

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$
$$\therefore \frac{d^n}{ds^n}F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{d^n}{ds^n}(e^{isx})dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot(ix)^n e^{isx}dx$$
$$[\because D^n(e^{as}) = a^n e^{as}]$$
$$= i^n\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f(x)\cdot x^n)e^{isx}dx$$

$$
\begin{aligned}
&= i^n F[x^n f(x)] \\
\therefore\ F[x^n f(x)] &= \frac{1}{i^n}\frac{d^n}{ds^n}F(s) \\
&= (-i)^n \frac{d^n}{ds^n}F(s)
\end{aligned}
$$

P.7. $F[f'(x)] = -isF(s)$ if $f(x) \to 0$ as $x \to \pm\infty$.

PROOF

$$
\begin{aligned}
F[f'(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\left\{\left[e^{isx}f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{isx}(is)\cdot f(x)dx\right\} \\
&= -isF(s).
\end{aligned}
$$

NOTE:

$$
\begin{aligned}
F[f^{(n)}(x)] = (-is)^n F(s) \text{ if } \quad f, f', f'', \ldots, f^{(n-1)} \\
\to 0 \text{ as } x \to \pm\infty.
\end{aligned}
$$

P.8. $F\left[\int_a^x f(x)dx\right] = \frac{i}{s}F(s)$.

PROOF

$$
\begin{aligned}
\text{Let } \phi(x) &= \int_a^x f(x)dx, \text{ then } \phi'(x) = f(x) \\
\text{By P.7.,} \quad F[\phi'(x)] &= -isF[\phi(x)] \\
\therefore \quad F[\phi(x)] &= \frac{1}{-is}F[\phi'(x)] = \frac{i}{s}F[f(x)] \\
&= \frac{i}{s}F(s)
\end{aligned}
$$

P.9. $F[f(-x)] = F(-s)$.

PROOF

$$\begin{aligned} F[f(-x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x) \cdot e^{isx} dx \quad \text{put } t = -x \\ &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t) \cdot e^{-ist} (-dt) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i(-s)t} dt \\ &= F(-s). \end{aligned}$$

NOTE:

(1) If $F[f(x)] = f(s)$, then the function $f(x)$ is called self reciprocal.

(2) If $f(t) = \begin{cases} e^{-kt} g(t), & t > 0 \\ 0, & t < 0 \end{cases},$

then $F[f(t)] = \frac{1}{\sqrt{2\pi}} L[g(t)].$

PROOF

$$\begin{aligned} F[f(t)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} + \int_{0}^{\infty} \right\} f(t) e^{ist} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-kt} g(t) \cdot e^{ist} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} g(t) e^{-(k-is)t} dt \quad \text{put } k - is = w \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} g(t) \cdot e^{-wt} dt \\ &= \frac{1}{\sqrt{2\pi}} L[g(t)]. \end{aligned}$$

4 FOURIER COSINE AND SINE INTEGRALS

In the study of Fourier series if a function $f(x)$ is defined in

$0 < x < l$, we extend it to $-l < x < l$ in such a way that $f(x)$ is periodic with period $2l$ and find its Fourier series. Thus we obtain the half range cosine series or half range sine series of

$f(x)$ depending upon the extension as an even function or an odd function. In the same way, if $f(x)$ is defined in $0 < x < \infty$, one can extend it to $-\infty < x < \infty$ either as an even function or as an odd function (depending upon the symmetries about the real axis or about the origin) and obtain the Fourier cosine transforms and Fourier sine transforms.

By Fourier integral theorem,

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]dw$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt dt \quad \text{and}$$
$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\, dt$$

If $f(x)$ is even in $-\infty < x < \infty$, then $f(x)\cos wx$ is also even but $f(x)\sin wx$ is odd.

So that $A(w) = \frac{2}{\pi}\int_0^\infty f(t)\cos wt dt$ and $B(w) = 0$.

Thus the above equation becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\cos wt dt\right)\cos wx dw. \tag{15}$$

Similarly, if $f(x)$ is odd in $-\infty < x < \infty$, then $f(x)\cos wx$ is odd and $f(x)\sin wx$ is even.

So that $A(w) = 0$ and $B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt dt$ and hence

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\sin wt\, dt\right)\sin wx dw \tag{16}$$

(8) is called Fourier cosine integral and (9) is called the Fourier sine integral.

5 FOURIER COSINE TRANSFORM AND ITS INVERSION FORMULA

Fourier cosine integral (8) can be written as

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\cos st\, dt\right)\cos sx ds \quad \text{(by replacing } w \text{ by } s\text{)} \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\,\cos st\, dt\right\}\cos sx ds \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(s)\cos sx ds \end{aligned} \tag{17}$$

where

$$\hat{f}_c(s) = \hat{f}_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\, dt. \tag{18}$$

The function $\hat{f}_c(s)$ as defined by (18), is called the Fourier cosine transform of $f(x)$. Also the function $f(x)$, as given by (17), is called the inversion formula for the Fourier cosine transform.

6 FOURIER SINE TRANSFORM AND ITS INVERSION FORMULA

Fourier sine integral (16) can be written as

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\sin st\,dt\right)\sin sx\,dx \quad \text{(by replacing } w \text{ by } s\text{)} \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\,dt\right\}\sin sx\,ds \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(s)\sin sx\,ds \end{aligned} \tag{19}$$

where

$$\hat{f}_s(s) = \hat{f}_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\,dt. \tag{20}$$

The function $\hat{f}_s(s)$, as defined by (20) is known as the Fourier sine transform of $f(x)$. Also the function $f(x)$, as given by (19) is known as the inversion formula for the Fourier sine transform.

6.1 PROPERTIES OF FOURIER COSINE AND SINE TRANSFORMS

7 LIST OF FORMULAE

1. Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cdot\cos w(t-x)dt\,dw.$$

2. Complex form of the Fourier integral is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)e^{iw(t-x)}dt\,dw.$$

3. Fourier cosine integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\cos wt\,dt\right)\cos wx\,dw.$$

4. Fourier sine integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\sin wt\,dt\right)\sin wx\,dw.$$

5. Fourier transform of $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

and its inversion formula is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} ds.$$

6. Fourier cosine transform of $f(x)$ is

$$\hat{f}_c[f(x)] = \hat{f}_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cdot \cos sx dx$$

and its inversion formula is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(s) \cdot \cos sx ds.$$

7. Fourier sine transform of $f(x)$ is

$$\hat{f}_s[f(x)] = \hat{f}_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

and its inversion formula is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(s) \cdot \sin sx ds.$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 3.9

Find the Fourier complex transform of

$$f(x) = \begin{cases} k & \text{in } |x| \leq l \\ 0 & \text{in } |x| > l \end{cases}$$

[K. U. 2002 November]

Solution

Fourier transform of $f(x)$ is given by

$$\begin{aligned} F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-l}^{l} k \cdot e^{isx} dx = \frac{k}{\sqrt{2\pi}} \cdot \left[\frac{e^{isx}}{is} \right]_{-l}^{l} \\ &= \frac{k}{\sqrt{2\pi}} \left[\frac{1}{is} \left(e^{isl} - e^{-isl} \right) \right] = \sqrt{\frac{2}{\pi}} \cdot \frac{k \cdot \sin sl}{s} \end{aligned}$$

EXAMPLE 3.10

Find the Fourier inverse transform of $e^{-s^2/4}$ [K.U. 2004 May]

Solution

By inversion formula for Fourier transform,

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2/4} (\cos sx - i \sin sx)\, ds \\ &= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} e^{-s^2/4} \cos sx\, ds \quad (\because e^{-s^2/4} \sin sx \text{ is odd}) \\ &= \sqrt{\frac{2}{\pi}} \cdot \int_{0}^{\infty} e^{-s^2/4} \cos sx\, ds = I(\text{say}) \end{aligned} \tag{1}$$

Now

$$\begin{aligned} \frac{dI}{dx} &= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-s^2/4} (-\sin sx) \cdot s\, ds \\ &= \sqrt{\frac{2}{\pi}} \cdot 2 \int_{0}^{\infty} (\sin sx) \left(-\frac{s}{2} e^{-s^2/4} ds \right) \\ &= 2\sqrt{\frac{2}{\pi}} \left\{ \left[(\sin sx) \left(e^{-s^2/4} \right) \right]_0^{\infty} - \int_{0}^{\infty} (\cos sx) x \cdot e^{-s^2/4} ds \right\} \end{aligned}$$

(Integrating by parts)

$$\begin{aligned} &= -2x \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-s^2/4} \cos sx\, ds \\ &= -2xI \end{aligned}$$

Seperating the variables,

$$\frac{dI}{I} = -2x dx \ \text{ or } \log I = -x^2 + \log A$$

$$\text{or } \ I = A e^{-x^2} \tag{2}$$

Now when $x = 0$, from (1)

$$\begin{aligned} I &= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-s^2/4} ds = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-y^2} 2dy \text{ where } y = s/2 \\ &= \sqrt{\frac{2}{\pi}} \cdot \sqrt{\pi} = \sqrt{2} \quad (\because 2 \int_{0}^{\infty} e^{-y^2} dy = \sqrt{\pi}) \end{aligned}$$

Also when $x = 0$, from (2), $I = A$

$$\therefore A = \sqrt{2}$$

(2) becomes $I = f(x) = \sqrt{2} e^{-x^2}$.

EXAMPLE 3.11

Find the complex Fourier transform of $e^{-a|x|}, a > 0$.

Solution

Fourier transform of $f(x)$ is given by

$$\begin{aligned}
F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i \sin sx)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} e^{-a|x|} \cos sx\, dx) \qquad (\because e^{-a|x|} \sin sx \text{ is odd}) \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax} \cos sx\, dx \\
&= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-ax}}{a^2+s^2} (-a \cos sx + s \sin sx) \right\}_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}
\end{aligned}$$

EXAMPLE 3.12

Find the Fourier transform of

$$f(x) = \begin{cases} a - |x| & \text{if } |x| < a \\ 0 & \text{if } |x| > a > 0. \end{cases}$$

Solution

Fourier transform of $f(x)$ is given by

$$\begin{aligned}
F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|)(\cos sx + i \sin sx) dx \\
&= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_{0}^{a} (a - |x|) \cos sx dx \qquad (\because (a - |x|) \sin sx \text{ is odd}) \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{a} (a - x) \cos sx\, dx \\
&= \sqrt{\frac{2}{\pi}} \left\{ (a - x) \left(\frac{\sin sx}{s} \right) - (-1) \left(\frac{-\cos sx}{s^2} \right) \right\}_0^{a}
\end{aligned}$$

$$= \sqrt{\frac{2}{\pi}}\left\{-\frac{\cos as}{s^2}+\frac{1}{s^2}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$$
$$= \sqrt{\frac{2}{\pi}}\cdot\frac{2\sin^2(as/2)}{s^2}$$

EXAMPLE 3.13

Find the Fourier cosine transform of e^{-5x} [K.U. 1998 November]

Solution

Fourier cosine transform of $f(x)$ is

$$\hat{f_c}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-5x}\cos sx\,dx$$
$$= \sqrt{\frac{2}{\pi}}\left\{\frac{e^{-5x}}{25+s^2}(-5\cos sx + s\sin sx)\right\}_0^\infty$$
$$= \sqrt{\frac{2}{\pi}}\left[\frac{5}{25+s^2}\right\}$$

EXAMPLE 3.14

Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

[K.U. 2000 April]

Solution

Fourier cosine transform of $f(x)$ is given by

$$\hat{f_c}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$
$$= \sqrt{\frac{2}{\pi}}\int_0^1 1\cdot\cos sx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_0^1$$
$$= \sqrt{\frac{2}{\pi}}\cdot\frac{\sin s}{s}$$

EXAMPLE 3.15

Find the Fourier cosine transform of e^{-x^2} [K.U. 2003 April]

Solution

Fourier cosine transform of $f(x) = e^{-x^2}$ is

$$\hat{f}_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cdot \cos sx \, dx = I \text{ (say)} \qquad (1)$$

Now

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} (-\sin sx) \cdot x \, dx$$
$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx (-2xe^{-x^2} dx)$$
$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left\{ \left[\sin sx \cdot e^{-x^2} \right]_0^\infty - \int_0^\infty s \cos sx \cdot e^{-x^2} \, dx \right\}$$
$$= \frac{1}{2}(-s) \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos sx \, dx$$
$$= -\frac{s}{2} I.$$

Seperating the variables,

$$\frac{dI}{I} = -\frac{s}{2} ds.$$

Integrating,

$$\log I = \frac{-s^2}{4} + \log A$$
$$\text{or } I = Ae^{-s^2/4} \qquad (2)$$

Now when $s = 0$, from (1),

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}.$$

Also when $s = 0$, from (2), $I = A$.

Equating, $A = \frac{1}{\sqrt{2}}$

$$\therefore I = \hat{f}_c(s) = \frac{1}{\sqrt{2}} e^{-s^2/4}.$$

EXAMPLE 3.16

Find the Fourier cosine transform of

$$f(x) = \sin x \text{ in } 0 < x < \pi$$

[K.U. 1998 April]

Solution

Fourier cosine transform of $f(x)$ is given by

$$\begin{aligned}
\hat{f}_c(s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cdot \cos sx\, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \cdot \cos sx\, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi \frac{1}{2}[\sin(s+1)x - \sin(s-1)x]\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{\cos(s+1)x}{s+1} + \frac{\cos(s-1)x}{s-1} \right\}_0^\pi \\
&= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{\cos(s+1)\pi}{s+1} + \frac{\cos(s-1)\pi}{s-1} + \frac{1}{s+1} - \frac{1}{s-1} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \frac{\cos s\pi}{s+1} - \frac{\cos s\pi}{s-1} + \frac{1}{s+1} - \frac{1}{s-1} \right\} \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{1+\cos s\pi}{1-s^2} \right)
\end{aligned}$$

EXAMPLE 3.17

Obtain the Fourier sine transform of $\frac{1}{x}$ [K. U. 1999 November]

Solution

$$\begin{aligned}
\hat{f}_s\left(\frac{1}{x}\right) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx\, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin t}{t/s} \cdot \frac{dt}{s} \text{ where } t = sx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin t}{t} dt = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} \left(\because \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \right) \\
&= \sqrt{\pi/2}
\end{aligned}$$

EXAMPLE 3.18

Find the Fourier sine transform of $e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{x\sin mx}{1+x^2}dx$.

[K.U. 2002 November]

Solution

$$\begin{aligned}
\hat{f}_s[e^{-|x|}] &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx.\\
&= \sqrt{\frac{2}{\pi}}\left\{\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx)\right\}_0^\infty\\
&= \sqrt{\frac{2}{\pi}}\frac{s}{1+s^2}
\end{aligned}$$

By inversion formula for Fourier sine transform

$$\begin{aligned}
f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(s)\cdot\sin sx\,ds\\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\cdot\frac{s}{1+s^2}\sin sx\,ds\\
&= \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{1+s^2}ds\\
\therefore \int_0^\infty \frac{s\sin sx}{1+s^2}ds &= \frac{\pi}{2}f(x) = \frac{\pi}{2}e^{-x}.
\end{aligned}$$

Replacing x by m, we get

$$\int_0^\infty \frac{s\sin ms}{1+s^2}ds = \frac{\pi}{2}e^{-m}.$$

Hence $\int_0^\infty \frac{x\sin mx}{1+x^2}dx = \frac{\pi}{2}e^{-m}$.

EXAMPLE 3.19

Find

(i) Fourier cosine transform of $\frac{1}{1+x^2}$ and

(ii) Fourier sine transform of $\frac{x}{1+x^2}$

[K.U. 1998 October/November]

Solution

$$\hat{f}_c\left[\frac{1}{1+x^2}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cdot\cos sx\cdot dx$$

Let

$$\int_0^\infty \frac{\cos sx}{1+x^2}dx=I \tag{1}$$

Then

$$\frac{dI}{ds}=\int_0^\infty -\frac{x\sin sx}{1+x^2}dx \tag{2}$$

$$=-\int_0^\infty \frac{x^2\sin sx}{x(1+x^2)}dx$$

$$=-\int_0^\infty \frac{[(1+x^2)-1]\sin sx}{x(1+x^2)}dx$$

$$=-\int_0^\infty \frac{\sin sx}{x}dx+\int_0^\infty \frac{\sin sx}{x(1+x^2)}dx$$

$$\therefore \frac{dI}{ds}=-\frac{\pi}{2}+\int_0^\infty \frac{\sin sx}{x(1+x^2)}dx \tag{3}$$

Now

$$\frac{d^2I}{ds^2}=\int_0^\infty \frac{x\cos sx}{x(1+x^2)}dx$$

$$=\int_0^\infty \frac{\cos sx}{1+x^2}dx=I$$

$$\therefore \frac{d^2I}{ds^2}-I=0 \quad \text{or } (D^2-1)I=0 \text{ where } D=\frac{d}{ds}$$

$$\therefore I=Ae^s+Be^{-s} \tag{4}$$

$$\frac{dI}{ds}=Ae^s-Be^{-s} \tag{5}$$

Putting $s=0$ in (1) and (4), we get

$$I=A+B=\int_0^\infty \frac{dx}{1+x^2}=(\tan^{-1}x)_0^\infty=\frac{\pi}{2} \tag{6}$$

Putting $s=0$ in (3) and (5), we get

$$\frac{dI}{ds}=A-B=-\frac{\pi}{2}. \tag{7}$$

Solving (**??**) and (12), we get $A=0$ and $B=\pi/2$. Substituting these in (4)

$$I=\frac{\pi}{2}e^{-s} \tag{8}$$

$$\therefore \quad \hat{f}_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{2}e^{-s} = \sqrt{\frac{\pi}{2}}e^{-s}$$

(ii) $$\hat{f}_s\left[\frac{x}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin sx}{1+x^2}dx$$
$$= \sqrt{\frac{2}{\pi}}\left(\frac{-dI}{dS}\right) \text{[by (2)]}$$
$$= -\sqrt{\frac{2}{\pi}}\left[-\frac{\pi}{2}e^{-s}\right], \text{ by using (8)}$$
$$= \sqrt{\frac{\pi}{2}}e^{-s}$$

EXAMPLE 3.20

Find the Fourier sine transform of $\dfrac{e^{-ax}}{x}, a > 0$.

Solution

$$\hat{f}_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx = I \text{ (say)} \qquad (1)$$
$$\text{Then } \frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}x\cos sx\,dx$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$
$$= \sqrt{\frac{\pi}{2}}\left\{\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right\}_0^\infty$$
$$= \sqrt{\frac{\pi}{2}}\cdot\frac{a}{a^2+s^2}.$$

Integrating wrt s, we get

$$I = \sqrt{\frac{2}{\pi}}\int \frac{a}{a^2+s^2}ds$$
$$= \sqrt{\frac{2}{\pi}}\cdot a\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + C$$
$$= \sqrt{\frac{2}{\pi}}\tan^{-1}(s/a) + C \qquad (2)$$

Put $s = 0$ in (1) and (2), we get

$$I = C = 0$$
$$\therefore I = \hat{f}_s(s) = \sqrt{\frac{2}{\pi}}\tan^{-1}(s/a).$$

EXAMPLE 3.21

Find the Fourier sine transform of

$$f(x) = \begin{cases} e^{-x}, & 0 \le x < b \\ 0, & x > b \end{cases}$$

[K. U. 2003 April]

Solution

$$\begin{aligned}
\hat{f}_s(s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^b e^{-x} \sin sx \, dx \\
&= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right\}_0^b \\
&= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{e^{-b}}{1+s^2} (-\sin sb - s \cos sb) \right] - \left[\frac{1}{1+s^2} (-s) \right] \right\} \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2} \left[s - e^{-b} (\sin sb + s \cos sb) \right]
\end{aligned}$$

EXAMPLE 3.22

Find the Fourier sine transform of

$$f(x) = \begin{cases} \sin x, & 0 \le x \le a \\ 0, & x > a \end{cases}$$

[K. U. 2003 November]

Solution

$$\begin{aligned}
\hat{f}_s(s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin x \cdot \sin sx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1}{2} [\cos(s-1)x - \cos(s+1)x] \, dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1} \right]_0^a \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1} \right]
\end{aligned}$$

EXERCISE 3.1

1. If $F(f(x)) = F(s)$, find $F[f(ax)]$ [K. U. 2000 April)

2. If $F(f(x)) = F(s)$, prove that $F[f(x-a)] = e^{ias}F(s)$

   [K.U. 2001 April)

3. Find the Fourier transform of the following functions

   (i) $f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < a \\ 0, & x > a \end{cases}$ [K.U. 2002 April]

   (ii) $f(x) = \begin{cases} x^2, & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a \end{cases}$

   (iii) $f(x) = xe^{-x}$ where $0 \leq x < \infty$

   (iv) $f(x) = \begin{cases} a^2 - x^2 & \text{if } |x| < a \\ 0 & \text{if } |x| \geq a \end{cases}$

   (v) $f(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1. \end{cases}$

   (vi) $f(x) = e^{-|x|}$

   (vii) $f(x) = \begin{cases} |x| & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$

   (viii) $f(x) = \begin{cases} \cos x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

   (ix) $f(x) = \begin{cases} e^{ikx} & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$

   (x) $f(x) = \begin{cases} \sin x & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a > 0. \end{cases}$

4. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| \geq a > 0. \end{cases}$$

Hence find

$$\text{(i)} \quad \int_0^\infty \frac{\sin x}{x} dx \text{ and}$$
$$\text{(ii)} \quad \int_{-\infty}^\infty \frac{\sin as \cdot \cos sx}{s} ds.$$

5. Find the Fourier integral representation of

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

6. Find the Fourier cosine transform of the following functions

   (i) $f(x) = e^{-2x} + 4e^{-3x}$ [K.U. 2001 October]

   (ii) $f(x) = e^{-2x} + 3e^{-x}$

   (iii) $f(x) = e^{-ax}/x$

   (iv) $f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$

   (v) $f(x) = e^{-ax}, a > 0$

   (vi) $f(x) = \begin{cases} x^2, & 0 < x \leq \pi \\ 0, & x > \pi \end{cases}$

7. Find the Fourier cosine transform of $f(x) = e^{-4x}$ and hence deduce that

   (i) $\int_0^\infty \frac{\cos 2x}{x^2 + 16} = \frac{\pi}{8} e^{-8}$ and

   (ii) $\int_0^\infty \frac{x \sin 2x}{x^2 + 16} = \frac{\pi}{2} e^{-8}$.

8. Find the Fourier sine transform of the following functions

   (i) $f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \\ 0 & \text{for } x \geq 2 \end{cases}$

   (ii) $f(x) = 5e^{-2x} + 2e^{-5x}$

(iii) $f(x) = \begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x \geq a \end{cases}$

(iv) $f(x) = e^{-ax}, a > 0$ and hence deduce that

$$\int_0^\infty \frac{s}{s^2+a^2} \sin sx\, ds = \frac{\pi}{2} e^{-ax}.$$

(v) $f(x) = \dfrac{x}{a^2+x^2}$

(vi) $f(x) = \begin{cases} x^2, & 0 < x \leq \pi \\ 0, & x > \pi \end{cases}$

(vii) $f(x) = e^{-3x} + 3e^{-2x}$.

(viii) $f(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$

9. Find the Fourier sine and cosine transforms of $\cosh x - \sinh x$.

10. Find $f(x)$ if its sine transform in e^{-as}.

11. Find $f(x)$ if its cosine transform is e^{-as}.

12. Show that $f(x) = xe^{-x^2/2}$ is self reciprocal with respect to Fourier sine transform.

ANSWERS

1. $\dfrac{1}{|a|} F(s/a)$.

3. (i) $\dfrac{1}{\sqrt{2\pi}} \left(\dfrac{e^{ias}-1}{is} \right)$

(ii) $\sqrt{\dfrac{2}{\pi}} \left[\dfrac{a^2 \sin(as)}{s} + \dfrac{2a\cos as}{s^2} - \dfrac{2\sin as}{s^3} \right]$

(iii) $\dfrac{1}{\sqrt{2\pi}} \left[\dfrac{1}{(1-is)^2} \right]$

(iv) $\dfrac{-4}{\sqrt{2\pi} s^3} (as \cos as - \sin as)$

(v) $\sqrt{\dfrac{2}{\pi}} \left(\dfrac{1-\cos s}{s^2} \right)$

(vi) $\sqrt{\dfrac{2}{\pi}} \left(\dfrac{1}{1+s^2} \right)$

(vii) $\sqrt{\frac{2}{\pi}}\left(\frac{as\sin as+\cos as-1}{s^2}\right)$

(viii) $\frac{1}{2\sqrt{2\pi}}\left[\frac{\sin(s+1)}{s+1}+\frac{(\sin s-1)}{s-1}+i\left(\frac{1-\cos(s+1)}{s+1}\right.\right.$
$\left.\left.+\frac{1-\cos(s-1)}{s-1}\right)\right]$

(ix) $\frac{i}{\sqrt{2\pi}}\left[e^{i(k+s)a}-e^{i(k+s)b}\right]/(k+s)$

(x) $\sqrt{\frac{2}{\pi}}\frac{i}{1-s^2}(s\sin a\ cosas-\cos a\sin as)$

4. $F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right)$ (i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{2}f(x)$

5. $f(x)=\frac{1}{\pi}\int_0^\infty\frac{w\sin wx+\cos wx}{1+w^2}dw$

6. (i) $\sqrt{\frac{2}{\pi}}\left[\frac{2}{s^2+4}+\frac{12}{s^2+9}\right]$ (ii) $\sqrt{\frac{2}{\pi}}\left[\frac{2}{s^2+4}+\frac{3}{s^2+1}\right]$

(iii) $-\frac{1}{\sqrt{2\pi}}\log(s^2+a^2)$ (iv) $\sqrt{\frac{2}{\pi}}\left[\frac{2\cos s}{s^2}-\frac{\cos 2s}{s^2}-\frac{1}{s^2}\right]$

(v) $\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$ (vi) $2\sqrt{2\pi}\cos(s\pi/s^2)$

7. $\hat{f}_c(s)=\sqrt{\frac{2}{\pi}}\left(\frac{4}{s^2+16}\right)$

8. (i) $\sqrt{\frac{2}{\pi}}\frac{2\sin s}{s^2}(1-\cos s)$

(ii) $\sqrt{\frac{2}{\pi}}\left[\frac{5s}{s^2+4}+\frac{2s}{s^2+25}\right]$

(ii) $\frac{1}{\sqrt{2\pi}}\left[\frac{2s}{s^2-1}-\left(\frac{\cos(s+1)a}{s+1}+\frac{\cos(s-1)a}{s-1}\right)\right]$

(iv) $\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$ (v) $\sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}e^{-as}\right]$

(vi) $\sqrt{\frac{2}{\pi}}\left[\frac{2}{s^3}(\cos s\pi-1)-\frac{\pi^2\cos s\pi}{s}\right]$

(vii) $\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+9}+\frac{3s}{s^2+4}\right]$

(viii) $\sqrt{\frac{2}{\pi}}\cdot\frac{\sin s\pi}{1-s^2}.$

9. $\hat{f}_c(s)=\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+s^2}\right)$ and $\hat{f}_s(s)=\sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right)$

10. $f(x)=\sqrt{\frac{2}{\pi}}\left[\frac{x}{x^2+a^2}\right]$

11. $f(x)=\sqrt{\frac{2}{\pi}}\left[\frac{a}{x^2+a^2}\right]$

REFERENCES

[1] Erwin Kreyszig, *Advanced Engineering Mathematics*, 10th Edition, Wiley-India

[2] Peter V. O' Neil, *Advanced Engineering Mathematics*, Thompson Publications, 2007

[3] M Greenberg, *Advanced Engineering Mathematics*, 2nd Edition, Prentice Hall